5-6 The Remainder and Factor Theorems

Use synthetic substitution to find *f* (4) and *f* (–2) for each function.

1. $f(x) = 2x^3 - 5x^2 - x + 14$

SOLUTION:

Divide the function by $x - 4$.

$$\begin{array}{r|rrrr} 4 & 2 & -5 & -1 & 14 \\ & & 8 & 12 & 44 \\ \hline & 2 & 3 & 11 & |58 \end{array}$$

The remainder is 58. Therefore, $f(4) = 58$.
Divide the function by $x + 2$.

$$\begin{array}{r|rrrr} -2 & 2 & -5 & -1 & 14 \\ & & -4 & 18 & -34 \\ \hline & 2 & -9 & 17 & |-20 \end{array}$$

The remainder is –20. Therefore, $f(-2) = -20$.

2. $f(x) = x^4 + 8x^3 + x^2 - 4x - 10$

SOLUTION:

Divide the function by $x - 4$.

$$\begin{array}{r|rrrrr} 4 & 1 & 8 & 1 & -4 & -10 \\ & & 4 & 48 & 196 & 768 \\ \hline & 1 & 12 & 49 & 192 & |758 \end{array}$$

The remainder is 758. Therefore, $f(4) = 758$.
Divide the function by $x + 2$.

$$\begin{array}{r|rrrrr} -2 & 1 & 8 & 1 & -4 & -10 \\ & & -2 & -12 & 22 & -36 \\ \hline & 1 & 6 & -11 & 18 & |-46 \end{array}$$

The remainder is –46. Therefore, $f(-2) = -46$.

3. **NATURE** The approximate number of bald eagle nesting pairs in the United States can be modeled by the function $P(x) = -0.16x^3 + 15.83x^2 - 154.15x + 1147.97$, where x is the number of years since 1970. About how many nesting pairs of bald eagles can be expected in 2018?

SOLUTION:

From 1970 to 2018, there are 48 years.
Find $f(48)$ using synthetic substitution.

$$\begin{array}{r|rrrr} 48 & -0.16 & 15.83 & -154.15 & 1147.97 \\ & & -7.68 & 391.2 & 11378.4 \\ \hline & -0.16 & 8.15 & 237.05 & \vert 12526.37 \end{array}$$

We can expect 12,526 nesting pairs of bald eagles in 2018.

Given a polynomial and one of its factors, find the remaining factors of the polynomial.

4. $x^3 - 6x^2 + 11x - 6$; $x - 1$

SOLUTION:

Divide $x^3 - 6x^2 + 11x - 6$ by $x - 1$.

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \vert 0 \end{array}$$

$$\begin{aligned} x^3 - 6x^2 + 11x - 6 &= (x^2 - 5x + 6)(x - 1) \\ &= (x - 3)(x - 2)(x - 1) \end{aligned}$$

So, the remaining factors of the polynomial are $x - 3$ and $x - 2$.

5. $x^3 + x^2 - 16x - 16$; $x + 1$

SOLUTION:

Divide $x^3 + x^2 - 16x - 16$ by $x + 1$.

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -16 & -16 \\ & & -1 & 0 & 16 \\ \hline & 1 & 0 & -16 & \vert 0 \end{array}$$

$$\begin{aligned} x^3 + x^2 - 16x - 16 &= (x^2 - 16)(x + 1) \\ &= (x + 4)(x - 4)(x + 1) \end{aligned}$$

So, the remaining factors of the polynomial are $x + 4$ and $x - 4$.

6. $3x^3 + 10x^2 - x - 12$; $x - 1$

SOLUTION:

Divide $3x^3 + 10x^2 - x - 12$ by $x - 1$.

$$\begin{array}{r|rrrr} 1 & 3 & 10 & -1 & -12 \\ & & 3 & 13 & 12 \\ \hline & 3 & 13 & 12 & |0 \end{array}$$

$$\begin{aligned} 3x^3 + 10x^2 - x - 12 &= (3x^2 + 13x + 12)(x - 1) \\ &= (x + 3)(3x + 4)(x - 1) \end{aligned}$$

So, the remaining factors of the polynomial are $x + 3$ and $3x + 4$.

7. $2x^3 - 5x^2 - 28x + 15$; $x + 3$

SOLUTION:

Divide $2x^3 - 5x^2 - 28x + 15$ by $x + 3$.

$$\begin{array}{r|rrrr} -3 & 2 & -5 & -28 & 15 \\ & & -6 & 33 & -15 \\ \hline & 2 & -11 & 5 & |0 \end{array}$$

$$\begin{aligned} 2x^3 - 5x^2 - 28x + 15 &= (2x^2 - 11x + 5)(x + 3) \\ &= (x - 5)(2x - 1)(x + 3) \end{aligned}$$

So, the remaining factors of the polynomial are $x - 5$ and $2x - 1$.

Use synthetic substitution to find $f(-5)$ and $f(2)$ for each function.

8. $f(x) = x^3 + 2x^2 - 3x + 1$

SOLUTION:

Divide the function by $x + 5$.

$$\begin{array}{r|rrrr} -5 & 1 & 2 & -3 & 1 \\ & & -5 & 15 & -60 \\ \hline & 1 & -3 & 12 & |-59 \end{array}$$

The remainder is –59. Therefore, $f(-5) = -59$.
Divide the function by $x - 2$.

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -3 & 1 \\ & & 2 & 8 & 10 \\ \hline & 1 & 4 & 5 & |11 \end{array}$$

The remainder is 11. Therefore, $f(2) = 11$.

9. $f(x) = x^2 - 8x + 6$

SOLUTION:

Divide the function by $x + 5$.

$$\begin{array}{r|rrr} -5 & 1 & -8 & 6 \\ & & -5 & 65 \\ \hline & 1 & -13 & |71 \end{array}$$

The remainder is 71. Therefore, $f(-5) = 71$.
Divide the function by $x - 2$.

$$\begin{array}{r|rrr} 2 & 1 & -8 & 6 \\ & & 2 & -12 \\ \hline & 1 & -6 & |-6 \end{array}$$

The remainder is –6. Therefore, $f(2) = -6$.

10. $f(x) = 3x^4 + x^3 - 2x^2 + x + 12$

SOLUTION:

Divide the function by $x + 5$.

–5	3	1	–2	1	12
		–15	70	–340	1695
	3	–14	68	–339	1707

The remainder is 1707. Therefore, $f(-5) = 1707$.
Divide the function by $x - 2$.

2	3	1	–2	1	12
		6	14	24	50
	3	7	12	25	62

The remainder is 62. Therefore, $f(2) = 62$.

11. $f(x) = 2x^3 - 8x^2 - 2x + 5$

SOLUTION:

Divide the function by $x + 5$.

–5	2	–8	–2	5
		–10	90	–440
	2	–18	88	–435

The remainder is –435. Therefore, $f(-5) = -435$.
Divide the function by $x - 2$.

2	2	–8	–2	5
		4	–8	–20
	2	–4	–10	–15

The remainder is –15. Therefore, $f(2) = -15$.

12. $f(x) = x^3 - 5x + 2$

SOLUTION:

Divide the function by $x + 5$.

$$\begin{array}{r|rrrr} -5 & 1 & 0 & -5 & 2 \\ & & -5 & 25 & -100 \\ \hline & 1 & -5 & 20 & |-98 \end{array}$$

The remainder is –98. Therefore, $f(-5) = -98$.
Divide the function by $x - 2$.

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -5 & 2 \\ & & 2 & 4 & -2 \\ \hline & 1 & 2 & -1 & |0 \end{array}$$

The remainder is 0. Therefore, $f(2) = 0$.

13. $f(x) = x^5 + 8x^3 + 2x - 15$

SOLUTION:

Divide the function by $x + 5$.

$$\begin{array}{r|rrrrrr} -5 & 1 & 0 & 8 & 0 & 2 & -15 \\ & & -5 & 25 & -165 & 825 & -4135 \\ \hline & 1 & -5 & 33 & -165 & 827 & |-4150 \end{array}$$

The remainder is –4150. Therefore, $f(-5) = -4150$.
Divide the function by $x - 2$.

$$\begin{array}{r|rrrrrr} 2 & 1 & 0 & 8 & 0 & 2 & -15 \\ & & 2 & 4 & 24 & 48 & 100 \\ \hline & 1 & 2 & 12 & 24 & 50 & |85 \end{array}$$

The remainder is 85. Therefore, $f(2) = 85$.

14. $f(x) = x^6 - 4x^4 + 3x^2 - 10$

SOLUTION:

Divide the function by $x + 5$.

–5	1	0	–4	0	3	0	–10
		–5	25	–105	525	–2640	13200
	1	–5	21	–105	528	–2640	13190

The remainder is 13190. Therefore, $f(-5) = 13{,}190$.
Divide the function by $x - 2$.

2	1	0	–4	0	3	0	–10
		2	4	0	0	6	12
	1	2	0	0	3	6	2

The remainder is 2. Therefore, $f(2) = 2$.

15. $f(x) = x^4 - 6x - 8$

SOLUTION:

Divide the function by $x + 5$.

–5	1	0	0	–6	–8
		–5	25	–125	655
	1	–5	25	–131	647

The remainder is –603. Therefore, $f(-5) = 647$.
Divide the function by $x - 2$.

2	1	0	0	–6	–8
		2	4	8	4
	1	2	4	2	–4

The remainder is –4. Therefore, $f(2) = -4$.

16. **FINANCIAL LITERACY** A specific car's fuel economy in miles per gallon can be approximated by $f(x) = 0.00000056x^4 - 0.000018x^3 - 0.016x^2 + 1.38x - 0.38$, where x represents the car's speed in miles per hour. Determine the fuel economy when the car is travelling 40, 50 and 60 miles per hour.

SOLUTION:

Find $f(40)$.

$$\begin{array}{r|rrrrr} 40 & 0.00000056 & -0.000018 & -0.016 & 1.38 & -0.38 \\ & & 0.0000224 & 0.000176 & -0.63296 & 29.8816 \\ \hline & 0.00000056 & 0.0000044 & -0.015824 & 0.74704 & |29.5016 \end{array}$$

Find $f(50)$.

$$\begin{array}{r|rrrrr} 50 & 0.00000056 & -0.000018 & -0.016 & 1.38 & -0.38 \\ & & 0.000028 & 0.0005 & -0.775 & 30.25 \\ \hline & 0.00000056 & 0.000011 & -0.0155 & 0.605 & |29.87 \end{array}$$

Find $f(60)$.

$$\begin{array}{r|rrrrr} 60 & 0.00000056 & -0.000018 & -0.016 & 1.38 & -0.38 \\ & & 0.0000336 & 0.000936 & -0.90384 & 28.5696 \\ \hline & 0.00000056 & 0.0000156 & -0.015064 & 0.47616 & |28.1896 \end{array}$$

The fuel economy when the car is travelling 40, 50 and 60 miles per hour are about 29.50 mpg, 29.87 mpg and 28.19 mpg respectively.

Given a polynomial and one of its factors, find the remaining factors of the polynomial.

17. $x^3 - 3x + 2$; $x + 2$

SOLUTION:

Divide $x^3 - 3x + 2$ by $x + 2$.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -3 & 2 \\ & & -2 & 4 & -2 \\ \hline & 1 & -2 & 1 & |0 \end{array}$$

$$\begin{aligned} x^3 - 3x + 2 &= (x^2 - 2x + 1)(x + 2) \\ &= (x - 1)^2 (x + 2) \end{aligned}$$

So, the remaining factor of the polynomial is $(x - 1)^2$.

18. $x^4 + 2x^3 - 8x - 16$; $x + 2$

SOLUTION:

Divide $x^4 + 2x^3 - 8x - 16$ by $x + 2$.

$$\begin{array}{r|rrrrr} -2 & 1 & 2 & 0 & -8 & -16 \\ & & -2 & 0 & 0 & 16 \\ \hline & 1 & 0 & 0 & -8 & |0 \end{array}$$

$$\begin{aligned} x^4 + 2x^3 - 8x - 16 &= (x^3 - 8)(x + 2) \\ &= (x - 2)(x^2 + 2x + 4)(x + 2) \end{aligned}$$

So, the remaining factors of the polynomial are $x - 2$ and $x^2 + 2x + 4$.

19. $x^3 - x^2 - 10x - 8$; $x + 2$

SOLUTION:

Divide $x^3 - x^2 - 10x - 8$ by $x + 2$.

$$\begin{array}{r|rrrr} -2 & 1 & -1 & -10 & -8 \\ & & -2 & 6 & 8 \\ \hline & 1 & -3 & -4 & |0 \end{array}$$

$$\begin{aligned} x^3 - x^2 - 10x - 8 &= (x^2 - 3x - 4)(x + 2) \\ &= (x - 4)(x + 1)(x + 2) \end{aligned}$$

So, the remaining factors of the polynomial are $x - 4$ and $x + 1$.

20. $x^3 - x^2 - 5x - 3$; $x - 3$

SOLUTION:

Divide $x^3 - x^2 - 5x - 3$ by $x + 2$.

$$\begin{array}{r|rrrr} 3 & 1 & -1 & -5 & -3 \\ & & 3 & 6 & 3 \\ \hline & 1 & 2 & 1 & |0 \end{array}$$

$$\begin{aligned} x^3 - x^2 - 5x - 3 &= (x^2 + 2x + 1)(x + 2) \\ &= (x + 1)^2(x + 2) \end{aligned}$$

So, the remaining factor of the polynomial is $(x + 1)^2$.

21. $2x^3 + 17x^2 + 23x - 42$; $x - 1$

SOLUTION:

Divide $2x^3 + 17x^2 + 23x - 42$ by $x - 1$.

$$\begin{array}{r|rrrr} 1 & 2 & 17 & 23 & -42 \\ & & 2 & 19 & 42 \\ \hline & 2 & 19 & 42 & |0 \end{array}$$

$$\begin{aligned} 2x^3 + 17x^2 + 23x - 42 &= (2x^2 + 19x + 42)(x - 1) \\ &= (x + 6)(2x + 7)(x - 1) \end{aligned}$$

So, the remaining factors of the polynomial are $x + 6$, $2x + 7$.

22. $2x^3 + 7x^2 - 53x - 28$; $x - 4$

SOLUTION:

Divide $2x^3 + 7x^2 - 53x - 28$ by $x - 4$.

$$\begin{array}{r|rrrr} 4 & 2 & 7 & -53 & -28 \\ & & 8 & 60 & 28 \\ \hline & 2 & 15 & 7 & |0 \end{array}$$

$$\begin{aligned} 2x^3 + 7x^2 - 53x - 28 &= (2x^2 + 15x + 7)(x - 4) \\ &= (x + 7)(2x + 1)(x - 4) \end{aligned}$$

So, the remaining factors of the polynomial are $x + 7$, $2x + 1$.

23. $x^4 + 2x^3 + 2x^2 - 2x - 3$; $x - 1$

SOLUTION:

Divide $x^4 + 2x^3 + 2x^2 - 2x - 3$ by $x - 1$.

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & 2 & -2 & -3 \\ & & 1 & 3 & 5 & 3 \\ \hline & 1 & 3 & 5 & 3 & |0 \end{array}$$

$$\begin{aligned} x^4 + 2x^3 + 2x^2 - 2x - 3 &= (x^3 + 3x^2 + 5x + 3)(x - 1) \\ &= (x + 1)(x^2 + 2x + 3)(x - 1) \end{aligned}$$

So, the remaining factors of the polynomial are $x + 1$ and $x^2 + 2x + 3$.

24. $x^3 + 2x^2 - x - 2$; $x + 2$

SOLUTION:

Divide $x^3 + 2x^2 - x - 2$ by $x + 2$.

$$\begin{array}{r|rrrr} -2 & 1 & 2 & -1 & -2 \\ & & -2 & 0 & 2 \\ \hline & 1 & 0 & -1 & |0 \end{array}$$

$$\begin{aligned} x^3 + 2x^2 - x - 2 &= (x^2 - 1)(x + 2) \\ &= (x + 1)(x - 1)(x + 2) \end{aligned}$$

So, the remaining factors of the polynomial are $x + 1$ and $x - 1$.

25. $6x^3 - 25x^2 + 2x + 8$; $2x + 1$

SOLUTION:

Divide $6x^3 - 25x^2 + 2x + 8$ by $2x + 1$.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 6 & -25 & 2 & 8 \\ & & -3 & 14 & -8 \\ \hline & 6 & -28 & 16 & |0 \end{array}$$

$$\begin{aligned} 6x^3 - 25x^2 + 2x + 8 &= (6x^2 - 28x + 16)(2x + 1) \\ &= 2(x - 4)(3x - 2)(2x + 1) \end{aligned}$$

So, the remaining factors of the polynomial are $x - 4$ and $3x - 2$.

26. $16x^5 - 32x^4 - 81x + 162$; $2x - 3$

SOLUTION:

Divide $16x^5 - 32x^4 - 81x + 162$ by $2x - 3$.

$$\begin{array}{r|rrrrrr} \frac{3}{2} & 16 & -32 & 0 & 0 & -81 & 162 \\ & & 24 & -12 & -18 & -27 & -162 \\ \hline & 16 & -8 & -12 & -18 & -108 & |0 \end{array}$$

$$\begin{aligned} 16x^5 - 32x^4 - 81x + 162 &= \left(16x^4 - 8x^3 - 12x^2 - 18x - 108\right)(2x - 3) \\ &= (x - 2)(2x + 3)\left(4x^2 + 9\right)(2x - 3) \end{aligned}$$

So, the remaining factors of the polynomial are $x - 2$, $2x + 3$ and $4x^2 + 9$.

27. **BOATING** A motor boat travelling against waves accelerates from a resting position. Suppose the speed of the boat in feet per second is given by the function $f(t) = -0.04t^4 + 0.8t^3 + 0.5t^2 - t$, where t is the time in seconds.
a. Find the speed of the boat at 1, 2, and 3 seconds.
b. It takes 6 seconds for the boat to travel between two buoys while it is accelerating. Use synthetic substitution to find $f(6)$ and explain what this means.

SOLUTION:

a. Find $f(1)$.

$$\begin{aligned} f(1) &= -0.04(1)^4 + 0.8(1)^3 + 0.5(1)^2 - 1 \\ &= -0.04 + 0.8 + 0.5 - 1 \\ &= 0.26 \end{aligned}$$

The speed of the boat at 1 second is 0.26 ft/s.
Find $f(2)$.

$$\begin{aligned} f(2) &= -0.04(2)^4 + 0.8(2)^3 + 0.5(2)^2 - 2 \\ &= -0.04(16) + 0.8(8) + 0.5(4) - 2 \\ &= -0.64 + 6.4 + 2 - 2 \\ &= 5.76 \end{aligned}$$

The speed of the boat at 2 seconds is 5.76 ft/s.
Find $f(3)$.

$$\begin{aligned} f(3) &= -0.04(3)^4 + 0.8(3)^3 + 0.5(3)^2 - 3 \\ &= -0.04(81) + 0.8(27) + 0.5(9) - 3 \\ &= -3.24 + 21.6 + 4.5 - 3 \\ &= 19.86 \end{aligned}$$

The speed of the boat at 3 seconds is 19.86 ft/s.

b. Find $f(6)$ using synthetic substitution.

$$\begin{array}{r|rrrrr} 6 & -0.04 & 0.8 & 0.5 & -1 & 0 \\ & 0 & -0.24 & 3.36 & 23.16 & 132.96 \\ \hline & -0.04 & 0.56 & 3.86 & 22.16 & |132.96 \end{array}$$

Thus, $f(6)$ = 132.96 ft/s. This means the boat is traveling at 132.96 ft/s when it passes the second buoy.

28. **CCSS REASONING** A company's sales, in millions of dollars, of consumer electronics can be modeled by $S(x) = -1.2x^3 + 18x^2 + 26.4x + 678$, where x is the number of years since 2005.

a. Use synthetic substitution to estimate the sales for 2017 and 2020.

b. Do you think this model is useful in estimating future sales? Explain.

SOLUTION:

a. Find $S(12)$ for sales for 2017:

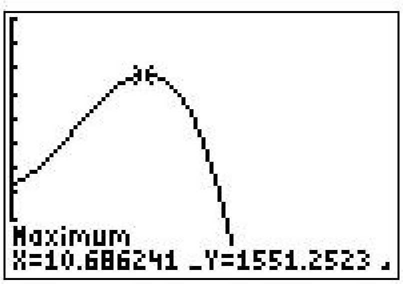
12	–1.2	18	26.4	678
	0	–14.4	43.2	835.2
	–1.2	3.6	69.9	1513.2

Find $S(15)$ for sales for 2020:

15	–1.2	18	26.4	678
	0	–18	0	396
	–1.2	0	26.4	1074

The sales for 2017 and 2020 are \$1513.2 million and \$1074 million respectively.

b.



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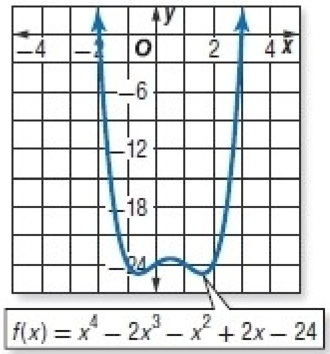
Sample answer: The graph of the function has a relative maximum at about $x = 11$ or the year 2016 and then the values rapidly decrease. This model could be useful for the next 15 years. After that, is is unlikely that the sales would decrease as rapidly as indicated.

Use the graphs to find all of the factors for each polynomial function.

29. 

SOLUTION:

The graph intersects the x-axis at $x = -2$ and $x = 3$. So, the factors are $(x + 2)$ and $(x - 3)$.

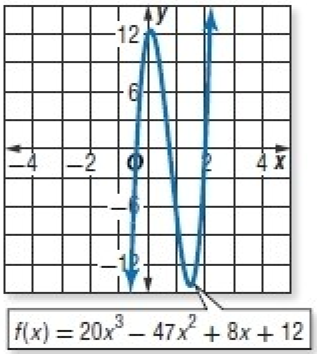
−2	1	−2	−1	2	−24
	0	−2	8	−14	24
	1	−4	7	−12	0

The depressed polynomial is $x^3 - 4x^2 + 7x - 12$.

3	1	−4	7	−12
	0	3	−3	12
	1	−1	4	0

The depressed polynomial is $\left(x^2 - x + 4\right)$.

The factors of the polynomial function are $x + 2$, $x - 3$, and $x^2 - x + 4$.

30. 

SOLUTION:

2	20	−47	8	12
	0	40	−14	−12
	20	−7	−6	0

The depressed polynomial is $\left(20x^2-7x-6\right)$.

$$\begin{aligned}20x^3-47x^2+8x+12&=(x-2)\left(20x^2-7x-6\right)\\&=(x-2)(4x-3)(5x+2)\end{aligned}$$

The factors of the polynomial function are $x - 2$, $4x - 3$, and $5x + 2$.

31. **MULTIPLE REPRESENTATIONS** In this problem, you will consider the function $f(x) = -9x^5 + 104x^4 - 249x^3 - 456x^2 + 828x + 432$.

a. ALGEBRAIC If $x - 6$ is a factor of the function, find the depressed polynomial.

b. TABULAR Make a table of values for $-5 \le x \le 6$ for the depressed polynomial.

c. ANALYTICAL What conclusions can you make about the locations of the other zeros based on the table? Explain your reasoning.

d. GRAPHICAL Graph the original function to confirm your conclusions.

SOLUTION:

a.

6	−9	104	−249	−456	828	432
		−54	300	306	−900	−432
	−9	50	51	−150	−72	0

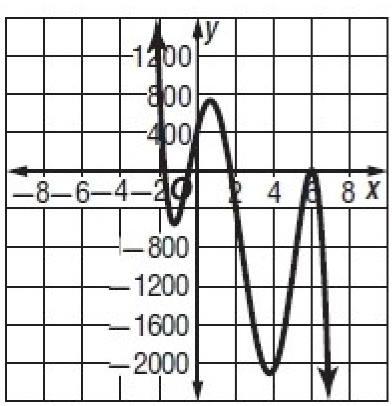
The depressed polynomial is $g(x) = -9x^4 + 50x^3 + 51x^2 - 150x - 72$.

b.

x	$g(x)$
–5	–9922
–4	–4160
–3	–1242
–2	–112
–1	70
0	–72
1	–130
2	88
3	558
4	1040
5	1078
6	0

c. There is a zero between x = –2 and x = –1 because $g(x)$ changes sign between the two values. There are also zeros between x = -1 and 0 and between and x = 1 and x = 2 because gx) changes sign between the two values. There is also a zero at x = 6.

d.



CCSS Find values of *k* so that each remainder is 3.

32. $(x^2 - x + k) \div (x - 1)$

SOLUTION:

Divide using synthetic division.

$$\begin{array}{r|rrr} 1 & 1 & -1 & k \\ & & 1 & 0 \\ \hline & 1 & 0 & 3 \end{array}$$

So, the value of k is 3.

33. $(x^2 + kx - 17) \div (x - 2)$

SOLUTION:
Divide using synthetic division.

$$\begin{array}{r|rrr} \underline{2} & 1 & k & -17 \\ & & 2 & 2k+4 \\ \hline & 1 & k+2 & |3 \end{array}$$

So, $2k + 4 - 17 = 3$.
Thus, $k = 8$.

34. $(x^2 + 5x + 7) \div (x + k)$

SOLUTION:
Divide using synthetic division.

$$\begin{array}{r|rrr} \underline{-k} & 1 & 5 & 7 \\ & & -k & -5k+k^2 \\ \hline & 1 & 5-k & |3 \end{array}$$

So, $k^2 - 5k + 7 = 3$.

$$k^2 - 5k + 4 = 0$$
$$(k-4)(k-1) = 0$$

Thus, $k = 1$ and $k = 4$.

35. $(x^3 + 4x^2 + x + k) \div (x + 2)$

SOLUTION:
Divide using synthetic division.

$$\begin{array}{r|rrrr} \underline{-2} & 1 & 4 & 1 & k \\ & & -2 & -4 & 6 \\ \hline & 1 & 2 & -3 & |3 \end{array}$$

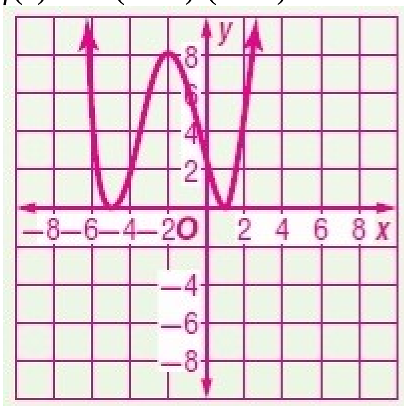
So, $k + 6 = 3$.
Thus, $k = -3$.

36. **OPEN ENDED** Write a polynomial function that has a double root of 1 and a double root of –5. Graph the function.

SOLUTION:

Sample answer: A polynomial function with 2 double roots is at least a 4th degree function. Even degree functions have end behavior in the same direction. Since the double roots are 1 and -5, the function will have $(x - 1)^2$ and $(x + 5)^2$ as factors.

$f(x) = 0.1(x – 1)^2(x + 5)^2$



CHALLENGE Find the solutions of each polynomial function.

37. $(x^2 – 4)^2 – (x^2 – 4) – 2 = 0$

SOLUTION:

$$\left(x^2-4\right)^2-\left(x^2-4\right)-2=0 \quad \text{Let } x^2-4=u.$$

$$u^2-u-2=0 \quad \text{Substitute.}$$

$$(u-2)(u+1)=0 \quad \text{Factor.}$$

$$u=2 \text{ or } u=-1$$

Substitute $x^2 – 4$ for u.

$$x^2-4=2 \qquad x^2-4=-1$$

$$x=\pm\sqrt{6} \text{ and } \qquad x=\pm\sqrt{3}$$

38. $(x^2 + 3)^2 - 7(x^2 + 3) + 12 = 0$

SOLUTION:

$$\left(x^2+3\right)^2-7\left(x^2+3\right)+12=0 \quad \text{Let } x^2+3=u.$$
$$u^2-7u+12=0 \quad \text{Substitute.}$$
$$(u-4)(u-3)=0 \quad \text{Factor.}$$
$$u=4 \text{ and } u=3$$

Substitute $x^2 + 3$ for u.

$$x^2+3=4 \quad \text{and} \quad x^2+3=3$$
$$x=1,-1 \text{ and } \quad x=0$$

39. **REASONING** Polynomial $f(x)$ is divided by $x - c$. What can you conclude if:

a. the remainder is 0?

b. the remainder is 1?

c. the quotient is 1, and the remainder is 0?

SOLUTION:

a. $x - c$ is a factor of $f(x)$.
b. $x - c$ is not a factor of $f(x)$.
c. $f(x) = x - c$

40. **CHALLENGE** Review the definition for the Factor Theorem. Provide a proof of the theorem.

SOLUTION:

If $x - a$ is a factor of $f(x)$, then $f(a)$ has a factor of $(a - a)$ or 0. Since a factor of $f(a)$ is 0, $f(a) = 0$. Now assume that $f(a) = 0$. If $f(a) = 0$, then the Remainder Theorem states that the remainder is 0 when $f(x)$ is divided by $x - a$. This means that $x - a$ is a factor of $f(x)$. This proves the Factor Theorem.

41. **OPEN ENDED** Write a cubic function that has a remainder of 8 for $f(2)$ and a remainder of –5 for $f(3)$.

SOLUTION:

Sample answer: Use synthetic substitution and guess and check to find a function that satisfies the criteria. First, try a function in which the coefficients are each +1.

2	1	1	1	?
		2	6	14
	1	3	7	8

Since the remainder is 8, the constant must be -6. Try this when the divisor is x +3.

3	1	1	1	-6
		3	12	39
	1	4	13	-5

Since the remainder does not equal -5, this cannot be the equation. Next, try $f(x) = -x^3 + x^2 + x + ?$.

2	-1	1	1	?
		-2	-2	-2
	-1	-1	-1	8

Since the remainder is 8, the constant must be 10. Try this when the divisor is $x - 5$.

3	-1	1	1	10
		-3	-6	-15
	-1	-2	-5	-5

This works so the function is $f(x) = -x^3 + x^2 + x + 10$.

42. **CHALLENGE** Show that the quadratic function $ax^4 + bx^3 + cx^2 + dx + e = 0$ will always have a rational root when the numbers 1, –2, 3, 4, and –6 are randomly assigned to replace a through f, and all of the numbers are used.

SOLUTION:

Sample answer: When $x = 1$, $f(1)$ is the sum of all of the coefficients and constants in $f(x)$, in this case, a, b, c, d, and e. The sum of a, b, c, d, and e is 0, so however the coefficients are arranged, $f(1)$ will always equal 0, and $f(x)$ will have a rational root.

43. **WRITING IN MATH** Explain how the zeros of a function can be located by using the Remainder Theorem and making a table of values for different input values and then comparing the remainders.

SOLUTION:

Sample answer: A zero can be located using the Remainder Theorem and a table of values by determining when the output, or remainder, is equal to zero. For instance, if $f(6)$ leaves a remainder of 2 and $f(7)$ leaves a remainder of –1, then you know that there is a zero between $x = 6$ and $x = 7$.

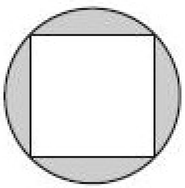
44. $27x^3 + y^3 =$
A $(3x + y)(3x + y)(3x + y)$
B $(3x + y)(9x^2 - 3xy + y^2)$
C $(3x - y)(9x^2 + 3xy + y^2)$
D $(3x - y)(9x^2 + 9xy + y^2)$

SOLUTION:

$$\begin{aligned}27x^3 + y^3 &= (3x)^3 + y^3 \\ &= (3x + y)(9x^2 - 3xy + y^2)\end{aligned}$$

So, B is the correct option.

45. **GRIDDED RESPONSE** In the figure, a square with side length $2\sqrt{2}$ is inscribed in a circle. The area of the circle is $k\pi$. What is the exact value of k?



SOLUTION:

Area of the circle is $k\pi$.

$$\begin{aligned}\text{So, } \pi r^2 &= k\pi \\ r^2 &= k\end{aligned}$$

The diagonal of the square is 4, thus the radius of the circle is 2.
Substitute 2 for r in the above equation and find k.

$$\begin{aligned}2^2 &= k \\ 4 &= k\end{aligned}$$

46. What is the product of the complex numbers $(4 + \boldsymbol{i})(4 - \boldsymbol{i})$?
F 15
G $16 - \boldsymbol{i}$
H 17
J $17 - 8\boldsymbol{i}$

SOLUTION:

$$\begin{aligned}(4+i)(4-i) &= 16 - 4i + 4i - i^2 && \text{Factor.} \\ &= 16 + 1 && \text{Simplify.} \\ &= 17 && \text{Add.}\end{aligned}$$

H is the correct option.

47. **SAT/ACT** The measure of the largest angle of a triangle is 14 less than twice the measure of the smallest angle. The third angle measure is 2 more than the measure of the smallest angle. What is the measure of the smallest angle?

A 46
B 48
C 50
D 52
E 82

SOLUTION:

Let the smallest angle be x.
So, the largest angle is $2x – 14$ and the third angle is $x + 2$.
The sum of the measures of angles of a triangle is 180.

$$
\begin{aligned}
x+2x-14+x+2 &= 180 && \text{Triangle Angle-Sum Theorem}\\
4x-12 &= 180 && \text{Combine like terms.}\\
4x &= 192 && \text{Add 12 to each side.}\\
x &= 48 && \text{Divide each side by 4.}
\end{aligned}
$$

The measure of the smallest angle is 48.
B is the correct option.

Solve each equation.

48. $x^4 – 4x^2 – 21 = 0$

SOLUTION:

$$
\begin{aligned}
x^4-4x^2-21 &= 0 &&\\
\left(x^2\right)^2-4\left(x^2\right)-21 &= 0 && \text{Let } x^2 = u.\\
u^2-4u-21 &= 0 && \text{Substitute.}\\
(u-7)(u+3) &= 0 && \text{Factor.}\\
u=7 \text{ and } u &= -3
\end{aligned}
$$

Substitute x^2 for u.

$$
\begin{aligned}
&x^2 = 7 \quad \text{and} \quad x^2 = -3\\
&x = \pm\sqrt{7} \text{ and } \quad x = \pm i\sqrt{3}
\end{aligned}
$$

49. $x^4 - 6x^2 = 27$

SOLUTION:

$$\begin{aligned} x^4 - 6x^2 - 27 &= 0 \\ \left(x^2\right)^2 - 6\left(x^2\right) - 27 &= 0 \quad \text{Let } x^2 = u. \\ u^2 - 6u - 27 &= 0 \quad \text{Substitute.} \\ (u-9)(u+3) &= 0 \quad \text{Factor.} \\ u = 9 \text{ and } u &= -3 \end{aligned}$$

Substitute x^2 for u.

$$\begin{aligned} x^2 &= 9 \quad \text{and} \quad x^2 = -3 \\ x &= \pm 3 \quad \text{and} \quad x = \pm i\sqrt{3} \end{aligned}$$

50. $4x^4 - 8x^2 - 96 = 0$

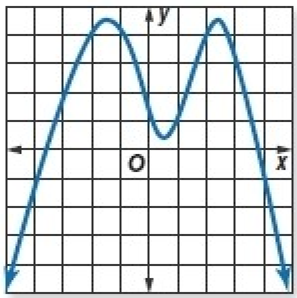
SOLUTION:

$$\begin{aligned} 4x^4 - 8x^2 - 96 &= 0 \\ \left(2x^2\right)^2 - 4\left(2x^2\right) - 96 &= 0 \quad \text{Let } 2x^2 = u. \\ u^2 - 4u - 96 &= 0 \quad \text{Substitute.} \\ (u-12)(u+8) &= 0 \quad \text{Factor.} \\ u = 12 \text{ and } u &= -8 \end{aligned}$$

Substitute $2x^2$ for u.

$$\begin{aligned} 2x^2 &= 12 \quad \text{and} \quad 2x^2 = -8 \\ x &= \pm\sqrt{6} \quad \text{and} \quad x = \pm 2i \end{aligned}$$

Complete each of the following.
a. Estimate the *x* coordinate of every turning point and determine if those coordinates are relative maxima or relative minima.
b. Estimate the *x*-coordinate of every zero.
c. Determine the smallest possible degree of the function.
d. Determine the domain and range of the function.

51. 

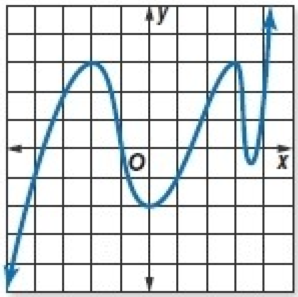
SOLUTION:

a. The value of $f(x)$ at about $x = -1.5$ is greater than the surrounding points, so there must be a relative maximum occur near $x = -1.5$.
The value of $f(x)$ at about $x = 2.5$ is greater than the surrounding points, so there must be a relative maximum occur near $x = 2.5$.
Therefore, the relative maximum occur at $x = -1.5$ and 2.5.
The value of $f(x)$ at about $x = 0.5$ is less than the surrounding points, so there must be a relative minimum near $x = 0.5$.

b. The graph intersects the *x*-axis at about $x = -3.5$ and at $x = 3.75$. The zeros of the function are at –3.5 and 3.75.

c. Since the graph has 3 turning points, the smallest possible degree of the function is 4.

d. $D = \{\text{all reals}\}$; $R = \{y \mid y \leq 4.5\}$.

52. 

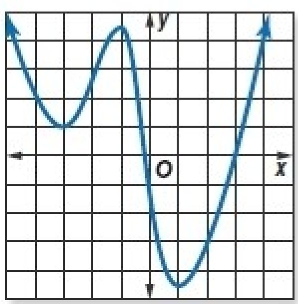
SOLUTION:

a. The value of $f(x)$ at about $x = –2$ is greater than the surrounding points, so there must be a relative maximum occur near $x = –2$.
The value of $f(x)$ at about $x = 3$ is greater than the surrounding points, so there must be a relative maximum occur near $x = 3$.
The value of $f(x)$ at about $x = 0$ is less than the surrounding points, so there must be a relative minimum near $x = 0$.
The value of $f(x)$ at about $x = 3.5$ is less than the surrounding points, so there must be a relative minimum near $x = 3.5$.

b. The graph intersects the x-axis at about –3.75, –1, 1.5, 3.25 and 3.75. Therefore, the zeros of the function are at –3.75, –1, 1.5, 3.25 and 3.75.

c. Since the graph has 4 turning points, the smallest possible degree of the function is 5.

d. $\mathrm{D} = \{\text{all reals}\}; \mathrm{R} = \{\text{all reals}\}$

53. 

SOLUTION:

a. The value of $f(x)$ at about $x = –1$ is greater than the surrounding points, so there must be a relative maximum occur near $x = –1$.
The value of $f(x)$ at about $x = –3$ is less than the surrounding points, so there must be relative minimum near $x = –3$.
The value of $f(x)$ at about $x = 1$ is less than the surrounding points, so there must be relative minimum near $x = 1$.

b. The graph intersects the x-axis at about –0.5 and 3. Therefore, the zeros of the function are at –0.5 and 3.

c. Since the graph has 3 turning points, the smallest possible degree of the function is 4.

d. $\mathrm{D} = \{\text{all real numbers}\}; \mathrm{R} = \{y \mid y \geq 4.5\}$

54. **HIGHWAY SAFETY** Engineers can use the formula $d = 0.05v^2 + 1.1v$ to estimate the minimum stopping distance d in feet for a vehicle traveling v miles per hour. If a car is able to stop after 125 feet, what is the fastest it could have been traveling when the driver first applied the brakes?

SOLUTION:

Substitute 125 for d in the formula and then use the Quadratic Formula to solve for v.

$0.05v^2 + 1.1v = 125$	Original equation
$0.05v^2 + 1.1v - 125 = 0$	Subtract 125 from each side.
$a = 0.05, b = 1.1$ and $c = -125$	
$v = \dfrac{-1.1 \pm \sqrt{(1.1)^2 - 4(0.05)(-125)}}{2(0.05)}$	Substitute into Quadratic Formula.
$= \dfrac{-1.1 \pm \sqrt{26.21}}{0.1}$	Simplify.
$\approx \dfrac{-1.1 \pm 5.12}{0.1}$	Simplify radical.
≈ 40.2 or -62.2	Simplify.

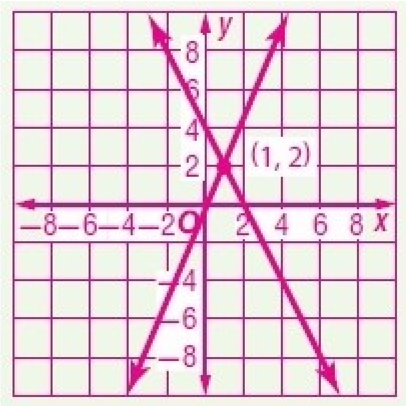
The car could travel about 40.2 mph, when the driver first applied the brakes.

Solve by graphing.

55. $y = 3x - 1$

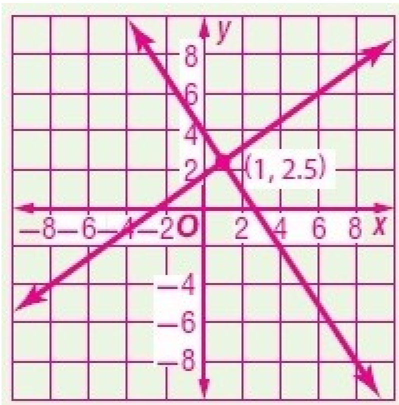
$y = -2x + 4$

SOLUTION:



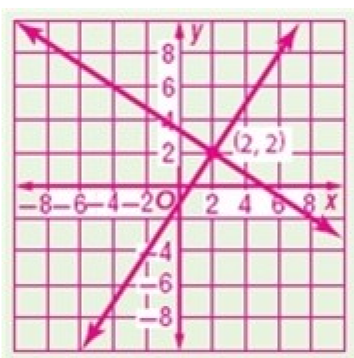
56. $3x + 2y = 8$
$-4x + 6y = 11$

SOLUTION:



57. $5x - 2y = 6$
$3x - 2y = 2$

SOLUTION:



If $c(x) = x^2 - 2x$ and $d(x) = 3x^2 - 6x + 4$, find each value.

58. $c(a + 2) - d(a - 4)$

SOLUTION:

$$\begin{aligned} c(a+2) &= (a+2)^2 - 2(a+2) \\ &= a^2 + 4a + 4 - 2a - 4 \\ &= a^2 + 2a \end{aligned}$$

$$\begin{aligned} d(a-4) &= 3(a-4)^2 - 6(a-4) + 4 \\ &= 3(a^2 - 8a + 16) - 6a + 24 + 4 \\ &= 3a^2 - 24a + 48 - 6a + 28 \\ &= 3a^2 - 30a + 76 \end{aligned}$$

Substitute $a^2 + 2a$ for $c(a + 2)$ and $3a^2 - 30a + 68$ for $d(a - 4)$ in the expression.

$$\begin{aligned} c(a+2) - d(a-4) &= a^2 + 2a - (3a^2 - 30a + 76) \\ &= a^2 + 2a - 3a^2 + 30a - 76 \\ &= -2a^2 + 32a - 76 \end{aligned}$$

59. $c(a - 3) + d(a + 1)$

SOLUTION:

$$\begin{aligned} c(a-3) &= (a-3)^2 - 2(a-3) \\ &= a^2 - 6a + 9 - 2a + 6 \\ &= a^2 - 8a + 15 \end{aligned}$$

$$\begin{aligned} d(a+1) &= 3(a+1)^2 - 6(a+1) + 4 \\ &= 3(a^2 + 2a + 1) - 6a - 6 + 4 \\ &= 3a^2 + 6a + 3 - 6a - 6 + 4 \\ &= 3a^2 + 1 \end{aligned}$$

Substitute $a^2 - 8a + 15$ for $c(a - 3)$ and $3a^2 + 1$ for $d(a + 1)$ in the expression.

$$\begin{aligned} c(a-3) + d(a+1) &= a^2 - 8a + 15 + 3a^2 + 1 \\ &= 4a^2 - 8a + 16 \end{aligned}$$

60. $c(-3a) + d(a + 4)$

SOLUTION:

$$\begin{aligned} c(-3a) &= (-3a)^2 - 2(-3a) \\ &= 9a^2 + 6a \end{aligned}$$

$$\begin{aligned} d(a+4) &= 3(a+4)^2 - 6(a+4) + 4 \\ &= 3\left(a^2 + 8a + 16\right) - 6a - 24 + 4 \\ &= 3a^2 + 24a + 48 - 6a - 20 \\ &= 3a^2 + 18a + 28 \end{aligned}$$

Substitute $9a^2 + 6a$ for $c(-3a)$ and $3a^2 + 18a + 28$ for $d(a + 4)$ in the expression.

$$\begin{aligned} c(-3a) + d(a+4) &= 9a^2 + 6a + 3a^2 + 18a + 28 \\ &= 12a^2 + 24a + 28 \end{aligned}$$

61. $3d(3a) - 2c(-a)$

SOLUTION:

$$\begin{aligned} c(-a) &= (-a)^2 - 2(-a) \\ &= a^2 + 2a \end{aligned}$$

$$\begin{aligned} d(3a) &= 3(3a)^2 - 6(3a) + 4 \\ &= 3\left(9a^2\right) - 18a + 4 \\ &= 27a^2 - 18a + 4 \end{aligned}$$

Substitute $a^2 + 2a$ for $c(-a)$ and $27a^2 - 18a + 4$ for $d(3a)$ in the expression.

$$\begin{aligned} 3d(3a) - 2c(-a) &= 3\left(27a^2 - 18a + 4\right) - 2\left(a^2 + 2a\right) \\ &= 81a^2 - 54a + 12 - 2a^2 - 4a \\ &= 79a^2 - 58a + 12 \end{aligned}$$

62. $c(a) + 5d(2a)$

SOLUTION:

$$c(a) = a^2 - 2a$$
$$\begin{aligned} d(2a) &= 3(2a)^2 - 6(2a) + 4 \\ &= 3(4a^2) - 12a + 4 \\ &= 12a^2 - 12a + 4 \end{aligned}$$

Substitute $a^2 - 2a$ for $c(a)$ and $12a^2 - 12a + 4$ for $d(2a)$ in the expression.

$$\begin{aligned} c(a) + 5d(2a) &= a^2 - 2a + 5(12a^2 - 12a + 4) \\ &= a^2 - 2a + 60a^2 - 60a + 20 \\ &= 61a^2 - 62a + 20 \end{aligned}$$

63. $-2d(2a + 3) - 4c(a^2 + 1)$

SOLUTION:

$$\begin{aligned} c(a^2+1) &= (a^2+1)^2 - 2(a^2+1) \\ &= a^4 + 2a^2 + 1 - 2a^2 - 2 \\ &= a^4 - 1 \end{aligned}$$
$$\begin{aligned} d(2a+3) &= 3(2a+3)^2 - 6(2a+3) + 4 \\ &= 3(4a^2 + 12a + 9) - 12a - 18 + 4 \\ &= 12a^2 + 36a + 27 - 12a - 18 + 4 \\ &= 12a^2 + 24a + 13 \end{aligned}$$

Substitute $a^4 - 1$ for $c(a^2 + 1)$ and $12a^2 + 24a + 13$ for $d(2a + 3)$ in the expression.

$$\begin{aligned} -2d(2a+3) - 4c(a^2+1) &= -2(12a^2 + 24a + 13) - 4(a^4 - 1) \\ &= -24a^2 - 48a - 26 - 4a^4 + 4 \\ &= -4a^4 - 24a^2 - 48a - 22 \end{aligned}$$